

The Complexity of Diffuse Reflections in a Simple Polygon

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Abstract. The complexity of the visibility region formed by a point light source after k diffuse reflections in a simple n -sided polygon is $O(n^9)$, which is the first result polynomial in n , with no dependence on k . This bound is an exponential improvement over the previous bound of $O(n^{2^{\lceil(k+1)/2\rceil+1}})$ due to Prasad et al. [8].

1 Introduction

Visibility problems in computational and combinatorial geometry have been studied extensively (see [3, 6, 9] and references therein). We confine our attention to results in the plane, more specifically those referring to visibility inside a simple polygon P with n vertices. Two points are *visible* to each other if the segment connecting them is contained in the polygon. The region visible from a point in P is a star-shaped polygon with at most n edges. The set of points of P visible from at least one point of a segment in P (the so-called “weak visibility polygon” from a segment) is a simple polygon with $O(n)$ edges and can be computed in linear time [5].

Aronov et al. [2, 1] and Davis [4] initiated the study of complexity of the region lit up by a single source of light in a simple polygon if reflection is allowed. Two models are considered. In both of them, any light incident upon a polygon corner is absorbed rather than reflected. In the *specular* reflection model, a light ray incident on a point in the interior of a polygon edge is reflected, as in geometric optics, with the angle of reflection equaling the angle of incidence. In the *diffuse* model which we consider in this paper, the light ray incident upon an interior point of an edge reflects in all possible interior directions. Aronov et al. [2] argue that for both diffuse and specular reflection the maximum complexity of the

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region lit up by a point light source with one reflection allowed is $\Theta(n^2)$. The results were generalized in [1] to any number k of reflections (where for simplicity we assume k is a constant and n can be arbitrarily large) and it was shown that for specular visibility this complexity is $O(n^{2k})$ and tight (at least for constant k). The case of multiple diffuse reflection is discussed by Prasad et al. [8], where they gave a bound of $O(n^{2^{\lceil (k+1)/2 \rceil + 1}})$ on the complexity of the region lit up by a point with at most k diffuse reflections. Surprisingly, even though this bound is exponential in k (for arbitrarily large n), no constructions were known for diffuse reflection with complexity $\omega(n^2)$, irrespective of the number of reflections used. This gave rise to the conjecture in [8] that this in fact is the correct answer, for $k \geq 1$ reflections. As the analysis in [2], among other things, proves that the region visible from a point with one diffuse reflection is always simply connected, it has been suggested that this remains true when more diffuse reflections are allowed. However, Pal [7] gives an example when this conjecture fails already when two reflections are allowed.

In this paper, we partially settle the former conjecture on multiple diffuse reflections, namely we argue that the complexity of the region visible from a point with at most k diffuse reflections is $O(n^9)$, for any value of k .

2 Main Result

We obtain the main result, Theorem 1, in the old-fashioned way by presenting a sequence of lemmas that slowly lead to the theorem. A fixed simple polygon P with n edges is implicit in all notation.

Definition 1 (Time). *By time k , we mean the state of the visible region after exactly k diffuse reflections.*

Definition 2 (Edge). *We use the term edge to refer exclusively to an entire edge of the polygon P . The letter e , and its sub-and-superscripted variants, always refers to an edge.*

Definition 3 (Initial visibility region). *Initially, one specified point light source p is illuminated in P . At time 0, point q is illuminated if the interior of the segment pq is interior to the polygon P .*

Definition 4 (Illumination by diffuse reflection). *If point p on an edge is illuminated at time k , then point q is illuminated at time $k + 1$ if the interior of the segment pq is interior to the polygon P . Points can only be illuminated in the manner described in the previous and current definitions.*

Definition 5 (Maximal illuminated segment). *We say segment x is maximally illuminated iff there does not exist an illuminated segment y such that $x \subset y$.*

Definition 6 (Triple). *We say (x, e, k) is illuminated if the maximally illuminated segment x on e is illuminated at time k .*

Definition 7 (Fundamental triple). We say (x, e, k) is a fundamental illumination, or $(x, e, k)^F$, if (x, e, k) is illuminated with the restriction that x is the first illuminated segment from either end of e at time k .

Definition 8 (Pentuple). We say (x, y, z, e, k) is illuminated iff x , y , and z are maximally illuminated, adjacent, and disjoint segments of e illuminated at time k . To avoid symmetry problems, we assume there is a clockwise total ordering on disjoint line segments on the polygon P from an arbitrary vertex on P , and that $x < y < z$ with regards to this ordering.

Definition 9 (Interior triple). We say (y, e, k) is a interior triple, or $(y, e, k)^I$, if (x, y, z, e, k) is a pentuple.

Lemma 1. The complexity of the illuminated regions on the boundary of the polygon at time k is at most the number of fundamentally illuminated regions $(x, e, k)^F$ plus the number of the interiorly illuminated regions $(y, e, k)^I$.

Proof. This is true since every triple must be either a fundamental triple or an interior triple, and there is only one illuminated region in either triple. Conversely, every segment that is lit has an associated triple.

2.1 Illuminations

Definition 10 (Illumination of triples, “ \rightarrow ” relation, defining light). We say (x, e, k) illuminates $(x', e', k + 1)$, or $(x, e, k) \rightarrow (x', e', k + 1)$ for short, if:

- $e \neq e'$.
- (x, e, k) and $(x', e', k + 1)$ are illuminated.
- If x was the only thing illuminated at time k , then at time $k + 1$ there would be a segment illuminated on edge e' , call it x'' , and $x'' \subseteq x'$. We call the light from x to x'' the defining light of $(x, e, k) \rightarrow (x', e', k + 1)$. (Verbal description: x illuminates either the whole maximally illuminated segment x' or one part of x' .)

Definition 11 (Interior illumination, “ \xrightarrow{I} ” relation). (See Figure 1.) We say that $(y, e, k)^I$ interior illuminates $(y', e', k + 1)^I$, or $(y, e, k) \xrightarrow{I} (y', e', k + 1)$ for short, iff

- $(y, e, k) \rightarrow (y', e', k + 1)$
- $\exists_{x, z, x', z'} (x, y, z, e, k)$ and $(x', y', z', e', k + 1)$ are pentuples. These are the defining pentuples of the interior illumination. (Note: This also implies (y, e, k) and $(y', e', k + 1)$ are interior triples.)
- $(x, e, k) \rightarrow (x'', e', k + 1)$ where $x'' \neq y'$ and x'' may or may not be the same as x'
- $(z, e, k) \rightarrow (z'', e', k + 1)$ where $z'' \neq y'$ and z'' may or may not be the same as z'

- *Verbal description:* An interior segment y on edge e illuminates another interior segment y' on edge e' at time $k + 1$. An adjacent segment of y , x , must also illuminate a segment x' on edge e' , but it is not necessary that the segment x' is an adjacent segment of y' . Another adjacent segment of y , z , must also illuminate a segment z' on edge e' , but it is not necessary that the segment z' is an adjacent segment of y' .

The defining light of an interior illumination is the union of the defining lights of the three illuminations used in the definition.

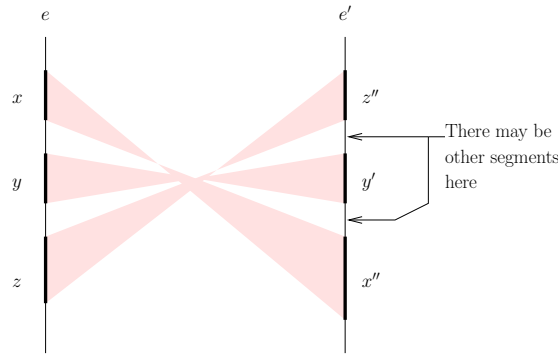


Fig. 1. An example of an interior illumination. The defining light of the illumination is the pink shaded region.

Definition 12 (“ $\overset{I}{\rightsquigarrow}$ ” relation). We use $\overset{I}{\rightsquigarrow}$ to represent the transitive closure of the $\overset{I}{\rightarrow}$ relation on interior triple. Thus, $a \overset{I}{\rightsquigarrow} b$ if there is a directed path from a to b in G^I .

Lemma 2. $(x, e, k) \not\overset{I}{\rightsquigarrow} (x', e, k')$ for all $k' \geq k + 2$. (Note: in this statement, and many others to follow, all variables are universally quantified unless otherwise noted.)

Proof. This is true because for every edge e there is always one point not on e that can see all of e . Such a point can be found by extending a ray from the line at a suitably small angle.

Lemma 3 (Complete illumination). At time $k = n$ the entire polygon is illuminated.

Proof. This would be trivial, if the corners of the polygon could be illuminated from an incident edge in one step. However, from the definition of illumination, this is not the case. It is trivial that if $k = 2n$ the entire polygon is illuminated, since there is always one point that is visible from two points on adjacent edges. We omit the proof that the polygon is illuminated at time $k = n$ since it is more

involved, and using the trivial $2n$ would not change any of the asymptotics of our results.

Note 1 (The end of time). From this point on we assume the time $k \leq n$, since beyond this time there is no additional complexity.

2.2 Graphs

Definition 13 (Graphs G , G^F , G^I). Define a directed graph G with vertices consisting of the union of the fundamental and interior triples, and edges defined by the “ \rightarrow ” relation as defined in definition 10. We also define the graphs G^F and G^I which is the subgraph of G induced by the nodes representing fundamental triples and interior triples, respectively. Edges in G^I are defined by definition 11. That is, a node $(x, e, k)^I$ has an outgoing edge to $(x', e', k + 1)^I$ in G^I iff $(x, e, k) \xrightarrow{I} (x', e', k + 1)$.

General idea: We first give an upper bound for the number of illuminated segments over all time and then we use this result to get an upper bound for the complexity of the visibility region. The total number of illuminated segments can be bounded by counting the number of nodes in G^F and G^I .

2.3 Bounding the Number of Fundamental Segments

Lemma 4. *There are at most $2n^2$ nodes in G^F .*

Proof. For a given e and k , there are at most 2 different segments x such that (x, e, k) is illuminated: Only the first segments from each end of e are fundamental triples. Since there are only n possible choices for e and the $k \leq n$ restriction of Note 1, this gives the result.

2.4 Bounding the Number of Interior Segments

Lemma 5 (Each interior segment can only illuminate n others). *Each illuminated segment can only illuminate n other segments. That is, for a given (x, e, k) there are only n segments y such that $(x, e, k) \rightarrow (y, e', k + 1)$.*

Proof. This follows from the observation that the complexity of the region illuminated by an edge is at most linear, with at most one segment of each polygon edge appearing on its boundary [5].

Definition 14 (Source node). *A source node in G^I is defined to be an interior triple, $(x, e, k)^I$, with in-degree 0 in G^I .*

Lemma 6 (Bounding the number of source nodes). *There are only $4n^3$ source nodes in G^I .*

Proof. In graph G , the parent of a source node (x, e, k) in G^I is either (1) a fundamental triple or (2) an interior triple, $(x', e', k - 1)^I$, with the restriction that $(x', e', k - 1) \not\stackrel{I}{\rightarrow} (x, e, k)$ in G^I .

There are at most $2n^3$ source nodes whose parent is a fundamental triple. This is true because there are at most $2n^2$ nodes in G^F (Lemma 4), each of which can illuminate n segments (Lemma 5). Each illuminated segment can appear in at most 1 interior triple of G^I .

On the other hand, there are at most $2(n - 1)(n^2)$ source nodes whose parent belongs to category 2 on all edges at all time. The harder observation is that all the illuminated segments on one edge can illuminate at most 2 source nodes whose parent belongs to category 2 on each edge at each time. The proof proceeds by contradiction.

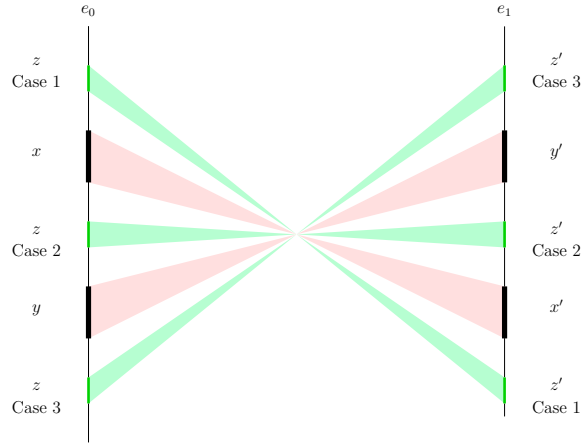


Fig. 2. No matter what the position of z is, if there are three illuminations between two edges, one must be interior.

Let (x, e_0, k) illuminate the source node $(x', e_1, k + 1)^I$ and (y, e_0, k) illuminate the source node $(y', e_1, k + 1)^I$. Assume there exists the third segment (z, e_0, k) which illuminates the source node $(z', e_1, k + 1)^I$. See Figure 2. If $x < z < y$, $(z', e - 1, k + 1)^I$ cannot be a source node in G^I by definitions 14 and 11. If $z < x < y$, $(x', e - 1, k + 1)^I$ cannot be a source node in G^I by definitions 14 and 11. If $x < y < z$, $(y', e - 1, k + 1)^I$ cannot be a source node in G^I by definitions 14 and 11. Therefore, all the illuminated segments on an edge can illuminate at most 2 source nodes whose parent belongs to category 2 on each edge at each time. Thus, all the illuminated segments on an edge can illuminate at most $2(n - 1)$ source nodes whose parent belongs to category 2 on all edges at each time. Since there are n edges, at most $2(n - 1)(n)$ source nodes have a parent whose belongs to category 2 at each time. This implies there are at most $2(n - 1)(n^2)$ source nodes whose parent belongs to category 2 on all edges at all time.

Therefore, there are $2n^3$ source nodes whose parent belongs to category 1 and $2(n-1)(n^2)$ source nodes whose parent belongs to category 2. Totally there are $2n^3 + 2(n-1)(n^2)$ source nodes in G^I . We simply say there are at most $4n^3$ sources nodes in G^I for simplicity.

Definition 15 (Chord c of an illumination: \xrightarrow{I}_c). We use the notion $(x, e_0, k) \xrightarrow{I}_c (x', e_1, k+1)$ to indicate $(x, e_0, k) \rightarrow (x', e_1, k+1)$ and that the defining light of this illumination passes through a chord c of the polygon.

Lemma 7 (Good chord c of an interior illumination: $\xrightarrow{I^g}_c$). For any $(y, e_0, k) \xrightarrow{I}_c (y', e_1, k+1)$ there is a chord c inside the polygon such that the endpoints of c are vertices of the polygon; one endpoint lies on the blue dashed line and another endpoint lies on the red dotted line. See Figure 3.

We call such a chord good, and use the notation $\xrightarrow{I^g}_c$ to indicate that c is good. To avoid symmetry problems, if there is more than 1 vertex on the dotted blue or dashed red line, we select the vertex which is closest to the illuminated segment, y , to be the endpoint of the good chord. This implies a good chord is uniquely defined for every interior illumination.

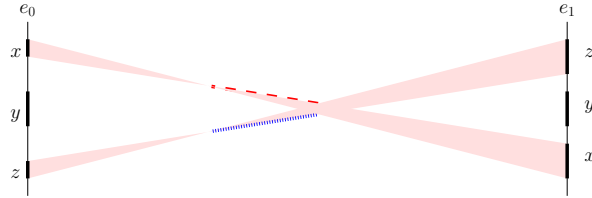


Fig. 3. If there are no polygon vertices on the boundary of the pink shaded region, z' or x' will be larger.

Proof. If no vertex lies on the blue line, the lower endpoint of the illuminated segment, z' , will have a different position. Similarly, if no vertex lies on the red line, the upper endpoint of the illuminated segment, x' , will have a different position. Therefore, a good chord must always exist for any $(y, e_0, k) \xrightarrow{I}_c (y', e_1, k+1)$

What is the purpose of a good chord? Through Lemma 8 to Lemma 11, We will prove that if a light passes through a good chord, it cannot go through the good chord in the opposite direction again in G^I . Therefore, a good chord will divide a simple polygon into two isolated regions.

Lemma 8. If $(y, e_0, k) \xrightarrow{I^g}_c (y', e_1, k+1)$ then for all $e_2 (y'', e_1, k+1) \not\xrightarrow{I}_c (y'', e_2, k+2)$.

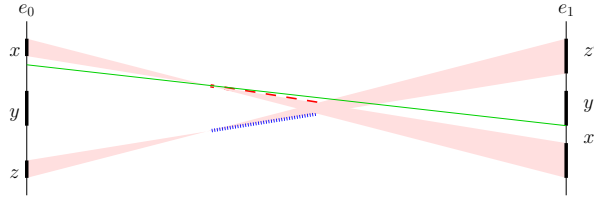


Fig. 4. The green shaded line goes from the bottom of y' to the left of the red dashed line. Its intersection with e_0 represents the highest point to the left of any chord c connecting the red dashed and blue dotted lines that can be illuminated by y' . Since the green shaded line is protected from the edges of the polygon by the pink shaded defining lights, it can never go to any edge above e_0 .

Proof. We start by noting that there must be a good chord c by Lemma 7. In Figure 4, the right endpoint of the green line is the lower endpoint of the illuminated segment, y' . It passes through the highest point on the red line. Based on the basic geometry concept, the left endpoint of the green line must be located below the illuminated segment, x . Since the endpoint of the good chord c is on the red line, y' cannot illuminate anything through c above the green line. Therefore, the illuminated segment, y'' , cannot be above the illuminated segment, x . This implies that e_2 cannot be above e_0 . By symmetry, we can conclude, e_2 must be the same as e_0 . However, e_0 is totally illuminated at time $k+2$ by Lemma 2. Therefore, for all e_2 , $(y', e_1, k + 1) \not\stackrel{I}{\rightarrow}_c (y'', e_2, k + 2)$.

Lemma 9 (On the intersection of chords). *For any two chords c and c' of the polygon P , if $(y, e, k) \stackrel{I^g}{\rightarrow}_{c'} (y', e', k + 1)$ and if c and c' intersect each other, then $(y, e, k) \stackrel{I}{\rightarrow}_c (y', e', k + 1)$.*

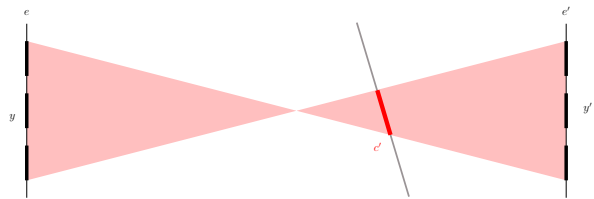


Fig. 5. If a chord c intersects c' , the defining light from y to y' must go through c . This is because no endpoint of c is allowed inside the pink shaded region.

Proof. Refer to Figure 5. The endpoints of c' must be on the the border of the defining light of $(y, e, k) \stackrel{I^g}{\rightarrow}_{c'} (y', e', k + 1)$ by definition. The pink shaded region(not including the boundary) must not contain any vertex. Since c intersects

with c' , if one endpoint of c is on the left side of the grey line above the pink shaded region, another point of c must be on the right side of the grey line below the pink shaded region. By symmetry, if one endpoint of c is on the left side of the grey line below the pink shaded region, another point of c must be on the right side of the grey line above the pink shaded region. In either case, $(y, e, k) \xrightarrow{I}_c (y', e', k + 1)$.

Lemma 10. For all $t \geq 1$, if $(y, e_0, k) \xrightarrow{I^g}_c (y', e_1, k + 1)$ and if for all $e_2 (y', e_1, k + 1) \not\xrightarrow{I}_c (y'', e_2, k + t)$ then for all $e_2 (y', e_1, k) \not\xrightarrow{I}_c (y''', e_2, k + t + 1)$.

Proof. Proof by contradiction. See Figure 6. Assume that for all $e_2, (y', e_1, k + 1) \not\xrightarrow{I}_c (y'', e_2, k + t)$ and $(y', e_1, k + 1) \xrightarrow{I} (y^5, e_5, k + t)$ and $(y^5, e_5, k + t) \xrightarrow{I}_c (y^7, e_7, k + t + 1)$. Let $(y', e_1, k + 1) \xrightarrow{I} (y^4, e_4, k + t - 1)$ and $(y^4, e_4, k + t - 1) \xrightarrow{I^g}_{c'} (y^5, e_5, k + t)$. The good chord c' must exist by Lemma 7. If c' intersects with c , then $(y^4, e_4, k + t - 1) \xrightarrow{I}_c (y^5, e_5, k + t)$ by Lemma 9. This is a contradiction to the assumption that $(y', e_1, k + 1) \not\xrightarrow{I}_c (y'', e_2, k + t)$ for all e_2 . If c' does not intersect with c , then e_0 and e_4 are on one side of the good chord c' and e_5 is on another side of c' . By Lemma 8, for all $e_6 (y^5, e_5, k + t) \not\xrightarrow{I}_{c'} (y^6, e_6, k + t + 1)$. It implies $(y^5, e_5, k + t) \not\xrightarrow{I}_c (y^7, e_7, k + t + 1)$. This is a contradiction to the assumption.

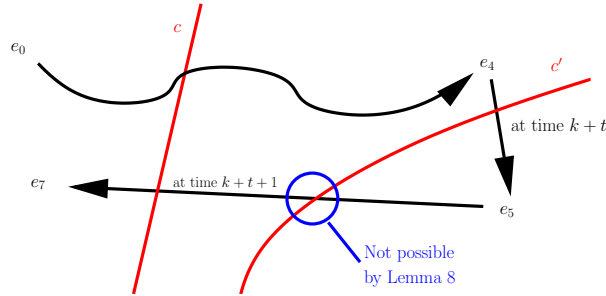


Fig. 6. The edge e_5 is on opposite sides of both c and c' from e_7 . Since c and c' do not cross, no edge can be interior illuminated from e_5 through chord c without going through c' also. Since e_5 was just illuminated through c' , the next illumination can not go through c' because of Lemma 8.

Lemma 11 (light cannot eventually go through the same chord twice).

If there exists a good chord such that $(y, e_0, k) \xrightarrow{I^g}_c (y', e_1, k + 1)$ then for all e_2, y'' and $k' > k + 1 (y', e_1, k) \not\xrightarrow{I}_c (y'', e_2, k')$.

Proof. The proof is by induction on k' , using the previous three lemmas.

The following lemma will prove that all good chords of interior illuminations from the same node in G^I are disjoint.

Lemma 12. *If $(y, e, k) \xrightarrow{I^g}_{c'} (y', e', k + 1)$ and $(y, e, k) \xrightarrow{I^g}_c (y'', e'', k + 1)$ then $(y, e, k) \not\xrightarrow{I}_{c'} (y', e', k + 1)$*

Proof. Assume $(y, e, k) \xrightarrow{I}_c (y', e', k + 1)$. Refer to Figure 7. The pink shaded region (not including the boundary) must not contain any vertex. Suppose both endpoints of c are on the boundary of the pink shaded region. This implies there are two good chords for the defining light of $(y, e, k) \xrightarrow{I} (y', e', k + 1)$. This can never happen by definition 7. Therefore, one endpoint of c must be above or below the pink shaded region. If that endpoint is on the right side of the grey line, it is not visible by (y, e, k) and thus, it cannot be an endpoint of the good chord c' by definition 7. This is a contradiction. If that endpoint is on the left side of the grey line, without loss of generality, we assume that endpoint is above the pink shaded region. Since the defining light of $(y, e, k) \xrightarrow{I} (y', e', k + 1)$ must pass through chord c , another endpoint of c must be on the lower boundary of the pink shaded region or below the pink shaded region. This implies at least one endpoint of c' will block the defining light of $(y, e, k) \xrightarrow{I^g}_c (y'', e'', k + 1)$. This is also a contradiction.

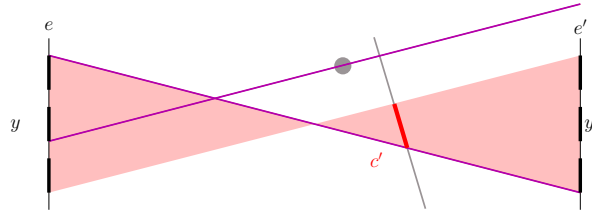


Fig. 7. If the grey point is an endpoint of c , then no light can originate on y , and pass through both the grey point and c' .

Lemma 13. *No two nodes reachable from the same source node in G^I have the same e value.*

Proof. By Definition 7, every edge in G^I must pass through a good chord. By Lemma 12, all good chords of interior illuminations from the same node in G^I are disjoint and the light that goes through one good chord does not go through the others. By Lemma 11, the light can never go through a good chord twice. Thus, a good chord divides the polygon into two isolated regions, and this process recurses (Figure 8(a)). There is no path between nodes in different isolated regions (Figure 8(b)). Therefore, for each source node $(y_0, e_0, k_0)^I$, there are no directed

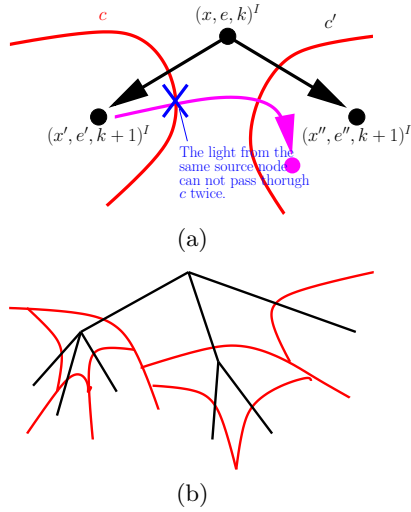


Fig. 8. Illustration of the proof of Lemma 13.

paths such that $(y_0, e_0, k_0) \xrightarrow{I} (y_3, e_3, k_3)$ and $(y_0, e_0, k_0) \xrightarrow{I} (y_4, e_3, k_4)$, when $y_3 \neq y_4$.

Lemma 14. *The graph G^I has at most $4n^4$ nodes.*

Proof. In Lemma 6 there are at most $4n^3$ sources in G^I . By Lemma 13, there are at most $n - 1$ nodes in G^I reachable from each of these sources.

2.5 The Complexity of the Visibility region

Lemma 15. *The total complexity of the illuminated edges over all time is at most $4n^4 + 2n^2$.*

Proof. Follows from Lemmas 1, 4, and 14.

Lemma 16. *If x segments are illuminated in a polygon with n edges at time k , then the complexity of the illuminated region of the polygon (including the interior) at time $k + 1$ is $O(nx^2)$.*

Proof. Since the region visible from one segment has complexity at most n [5], the intersection of x such regions is trivially $(xn)^2$. By observing that any region visible from one segment will intersect any segment not exterior to the polygon in exactly one place, this can be reduced to $O(nx^2)$.

Theorem 1. *The total complexity of the illuminated region at time k is $O(n^9)$.*

Proof. Lemma 15 gives a bound of $O(n^4)$ for the number of edges illuminated at time $k - 1$. By applying Lemma 16, the $O(n^9)$ bound is obtained.

Theorem 2 ([8]). *The total complexity of the illuminated region at time k is $\Omega(n^2)$.*

Conjecture 1 ([8]). The total complexity of the illuminated region at time k is $\Theta(n^2)$.

We still believe this conjecture holds. Our proof over-counts in myriad ways and surely is not tight.

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